THEORY OF HYPERSURFACES OF A FINSLER SPACE WITH THE GENERALIZED SQUARE METRIC

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Abstract

The emergence of generalized square metrics in Finsler geometry can be attributed to various classification concerning (α, β)-metrics. They have excellent geometric properties in Finsler geometry. Within the scope of this research paper, we have conducted an investigation into the generalized square metric denoted as $F(x,y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$, focusing specifically on its application to the Finslerian hypersurface. Furthermore, the classification and existence of first, second, and third kind of hyperplanes of the Finsler manifold has been established.

AMS Subject Classification: 53B40, 53C60

Key Words and Phrases: Finsler Space, Finsler metric, Generalized square metric, Hypersurface of Finsler space, Hyperplane.

1 Introduction

Let M be an n-dimensional differential manifold. Define a Finsler metric on the differentiable manifold M. This Finsler metric is known as Finsler fundamental fuction on the manifold M. Let us first define what is exactly mean by Finsler fundamental function.

Definition 1.1 (Finsler metric)**.** We say a function $F : TM \to R$ is a Finsler metric or Finsler fundamental function on the manifold M if F satisfy the following conditions:

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1. F is C^∞ away from zero vectors of the tangent spaces:
   That is, F is smooth on $TM\backslash\{0\} = \{(x, y)|x \in M, y \in T_xM, y \neq 0\}$.
   The smoothness property is desired so that we can apply differential calculus on the Finsler metric F.

2. Positivity of function F:
   $F(x, y) \geq 0$ for $x \in M$ and all $y \in T_xM$.
   This property ensures that the length of a tangent vector $y \in T_xM$ is either positive or zero. In terms of arc length this property also ensures that arc length defined by the integral 1.1 is either positive or zero.

3. Positive homogeneity of function F:
   $F(x, \lambda y) = \lambda F(x, y)$, $\forall\ \lambda > 0$; for $x \in M$ and all $y \in T_xM$.
   That is, F is +ve 1-homogeneous of first degree in the directional argument y. This property ensures that, length of a tangent vector $\lambda y \in T_xM$ is nothing but $\lambda \times F(x, y) \in R$.

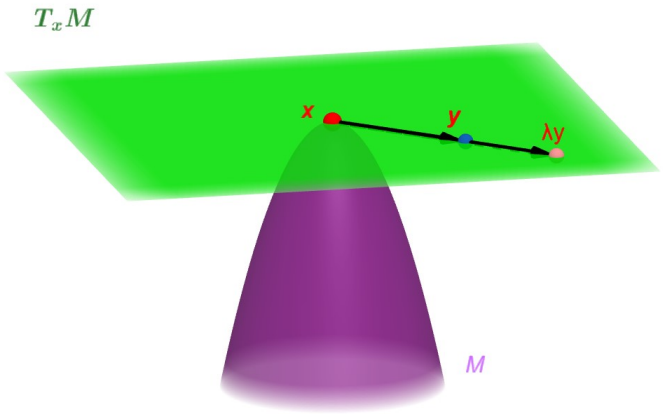

Figure 1: This figure shows the tangent vector $\lambda y \in T_xM$, $\lambda > 0$, is λ times the tangent vector $y \in T_xM$.

4. Strict convexity of the function F:
   $F : TM \to R$ is strictly convex over the tangent bundle TM.

It is important to note that strictly convex condition is equivalent to the hessian matrix $[g_{ij}]$, where $i, j \in \{1, 2, 3, ...dim(M)\}$, defined by $\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) = g_{ij}(x, y)$ is positive definite for any $(x, y) \in TM$. It is the convexity condition on the Finsler metric F that guaranties for the arc length minimization,

given by the following formula, of the admissible curves $\gamma : [a, b] \to M$ belonging to the set $C^{\infty}[a, b]$

$$s[\gamma(t)] = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt \tag{1.1}$$

can be achieved.

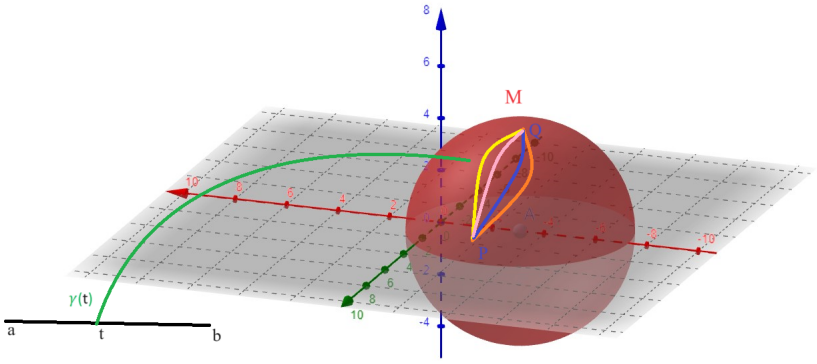

Figure 2: This figure shows the curves $\gamma : [a, b] \to M$ over the manifold M such that $\gamma(a) = P$ and $\gamma(b) = Q$.

In other words, the convexity condition on the Finsler metric is a geometric requirement that makes sure that the length of a curve can be defined and that the arc length functional $s[\gamma(t)]$ is well-behaved, particularly for the purpose of minimizing or finding geodesics with respect to the given Finsler metric F. Without convexity, the concept of length and the corresponding optimization problems will not make sense or may not have unique solutions.

In the definition provided above, $F(x, y)$ signifies the magnitude of the vector y within the tangent space T_xM originating from an arbitrary point x within the manifold M. This quantity is commonly referred to as the "F-length" or simply the "Finslerian length" of the tangent vector y in T_xM. If we fix the point $x \in M$ in the manifold M, then $F(x, .)$ can eat every tangent vector $y \in T_xM$ and spits a real number.
We can think of while considering physical problems involving position and direction, the Finsler metric $F(x, y)$ as having two arguments x and y representing location and velocity at any point $p \in M$ respectively.

Definition 1.2. (Finsler Manifold)
A differentiable manifold denoted as M, when equipped with a Finsler metric $F(x, y)$, is referred to as a Finsler manifold or Finsler space. This is typically denoted as (M, F).

Definition 1.3. (Finsler Geometry)
Consider a Finsler manifold denoted as (M, F), where $F(x, y)$ represents a Finsler metric. The branch of geometry that relies on the Finsler metric $F(x, y)$ defined on the manifold M is termed Finsler geometry.

In Finsler geometry, a significant and distinct class of Finsler metric known as the (α, β)-metric is of particular importance. This metric is defined as follows:

Definition 1.4 ((α, β)-metric)**.** Consider a Finsler space denoted as $(M, F(x, y))$, where $F(x, y)$ represents the Finsler fundamental function. This space is said to possess an (α, β)-metric if the fundamental function $F(x, y)$ can be expressed in the following manner:

$$F(x, y) = F\left(\alpha(x, y), \beta(x, y)\right)$$

In this expression, $F\left(\alpha(x, y), \beta(x, y)\right)$ is a differentiable function of two variables $\alpha(x, y)$ and $\beta(x, y)$. Here, $\alpha(x, y)$ is the Riemannian fundamental function defined as $\sqrt{a_{ij}(x)y^i y^j}$, where $a_{ij}(x)$ represents a Riemannian metric tensor, and $\beta(x, y)$ is a differential 1-form defined on the tangent bundle TM, with $b_i(x)$ representing a covariant vector field.

The class of (α, β)-metrics was originally introduced by the renowned geometer M. Matsumoto [9].

Some important examples of (α, β)-metric are:

Example 1.5. *The metric defined by*

$$F(x, y) = \alpha(x, y) + \beta(x, y)$$

is called Randers metric and the space $(M, F(x, y) = \alpha(x, y) + \beta(x, y))$ constructed with Randers metric is called Randers space. This metric was first introduced by Physicist G. Randers in 1941 [14], in his study of general relativity.

Example 1.6. *The metric defined by*

$$F = \frac{\alpha^2(x,y)}{\beta(x,y)}, \beta > 0$$

is called Kropina metric. A space $\left(M, F = \frac{\alpha^2(x,y)}{\beta(x,y)}\right)$ *constructed with Kropina metric is called Kropina space. This metric was introduced by the Russian physicist V.K. Kropina [7]. It has many important and interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics, relativistic field theory, control theory, evolution and developmental biology [11].*

Example 1.7. *The metric defined by*

$$F = \frac{\alpha^{n+1}(x,y)}{\beta^n(x,y)}, \beta > 0$$

is called generalized Kropina metric. A space $\left(M, F = \frac{\alpha^{n+1}(x,y)}{\beta^n(x,y)}\right)$ *constructed with Kropina metric is called generalized Kropina space.*

Example 1.8. *The metric defined by*

$$F = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)}, \alpha - \beta > 0$$

is called Matsumoto metric. A space $\left(M, F = \frac{\alpha^2(x,y)}{\alpha(x,y)-\beta(x,y)}\right)$ *constructed with Matsumoto metric is called Matsumoto space. This metric was first introduced by M. Matsumoto [8] while investigating the model of a Finsler space. This metric is also named as slope metric.*

Example 1.9. *The metric defined by*

$$F(x,y) = \frac{[\alpha(x,y) + \beta(x,y)]^2}{\alpha(x,y)}$$

is called square metric. We say the space $\left(M, F(x,y) = \frac{[\alpha(x,y)+\beta(x,y)]^2}{\alpha(x,y)}\right)$ *constructed with square metric Shen's square space.*

Example 1.10. *The metric defined by*

$$F(x,y) = \frac{[\alpha(x,y) + \beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$$

is called generalized square metric. We say the space $\left(M, F(x,y) = \frac{[\alpha(x,y)+\beta(x,y)]^{p+1}}{[\alpha(x,y)]^p}\right)$ *constructed with generalized square metric the1.generalized square space or the Shen's generalized square space. If we put* $n = 1$ *in above metric, i.e., in generalized square metric, we get the square metric of definition* (1.9).

Among these Finsler metrics this was the "generalized square metric" that draws our attention to work with hypersurface of a Finsler space $(M, F(x, y))$. Let us first define the meaning of a hypersurface:

Definition 1.11. A hypersurface is an embedded submanifold of codimention 1. That is, a submanifold of dimension less that 1 of a given manifold is called hypersuraface of the underlying manifold.

This means, if we have a manifold M such that dimension of M is n, then the submanifold having dimension $n-1$ will be understood as hypersurface of the given manifold.
A submanifold of dimension $n-1$ is generally denoted by the symbol M^{n-1}. In this paper, we will also denote a hupersurafce of a manifold M by the symbol M^{n-1}.

Example 1.12. *Let $M = R^2$ be a manifold. Then a straight line in Figure 3 denoted by M^{n-1} and defined by linear equation in two variables x and y, i.e., $ax + by + c = 0$ is a hypersuarface of the underlying manifold $M = R^2$, because dim(M)=dim(R^2)=2 while the dim(M^{n-1})=$n - 1 = 2 - 1 = 1$.*

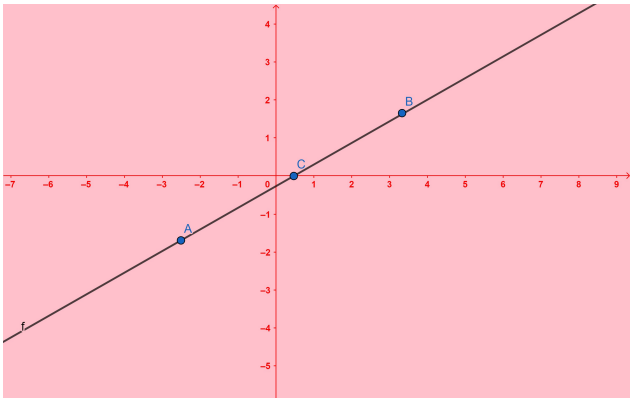

Figure 3: The line represented by eqution ax+by+c=0 is a hypersurface of the 2-dimensional manifold R^2.

Example 1.13. *Let $M = R^3$ be a manifold. Then the unit sphere in Figure 4 denoted by M^{n-1} and defined by $x^2 + y^2 + z^2 = 1$ is a hypersuarface of the underlying manifold $M = R^3$, because dim(M)=dim(R^3)=3 while the dim(M^{n-1})=$n - 1 = 3 - 1 = 2$.*

Matsumoto [10], a prominent Finslerian, was the first person who studies the hypersurfaces and charecterised the special hypersurfaces M^{n-1} of a Finsler manifold. He, specifically, characterised the properties of hypersurface M^{n-1} of Randers space [14]. After this, the number of Finslerians dramatically increased to show their interest in Finsler hypersurface M^{n-1}.

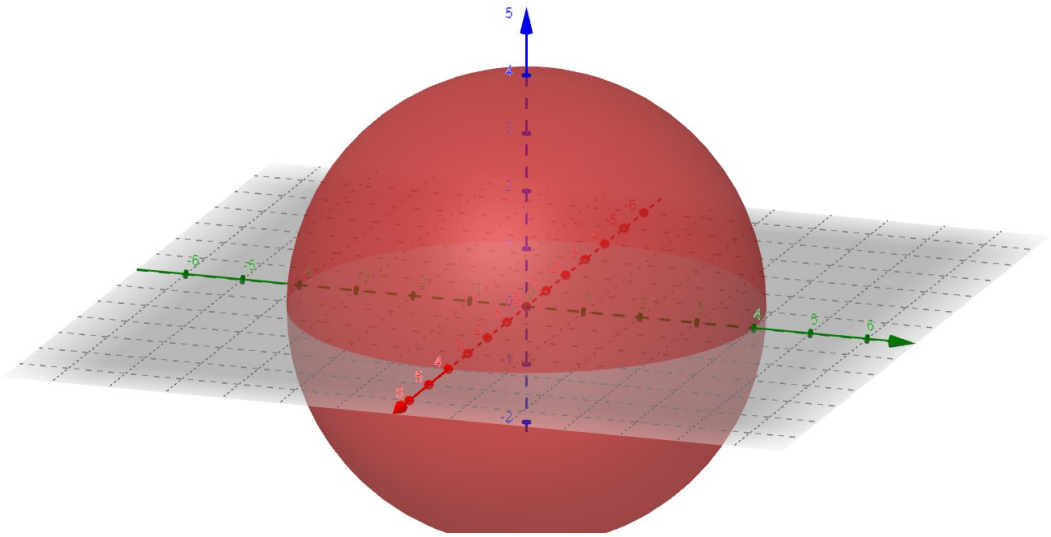

Figure 4: The sphere represented by equation $x^2 + y^2 + z^2 = 16$ is a hypersurface of the 3-dimensional manifold R^3.

Many authors around the world ([6], [5], [3], [15], [16], [18], [2], [12], [1], [4]) did study the properties of special hypersurface M^{n-1} and derived the conditions under which a Finsler hypersurface M^{n-1} of a Finsler manifold $(M, F(x, y))$ becomes a hyperplane of first kind, second kind but not of the third kind. Aim of the present paper is to investigate the hypersurface M^{n-1} of Finsler space using generalized square metric $F(x, y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$.

2 Preliminaries

We consider the Finsler space (M, F), where F is the generalized square metric, that is given by

$$F(\alpha, \beta) = \frac{(\alpha + \beta)^{n+1}}{\alpha^n} \tag{2.1}$$

Calculate all the partial derivatives of equation (2.1) up to second order,

we get

$$F_\alpha = \frac{(\alpha - n\beta)(\alpha+\beta)^n}{\alpha^{n+1}} \tag{2.2}$$

$$F_\beta = \frac{(n+1)(\alpha+\beta)^n}{\alpha^n} \tag{2.3}$$

$$F_{\alpha\alpha} = \frac{n(n+1)\beta^2(\alpha+\beta)^{n-1}}{\alpha^{n+2}} \tag{2.4}$$

$$F_{\beta\beta} = \frac{n(n+1)(\alpha+\beta)^{n-1}}{\alpha^n} \tag{2.5}$$

$$F_{\alpha\beta} = -\frac{n(n+1)\beta(\alpha+\beta)^{n-1}}{\alpha^{n+1}} \tag{2.6}$$

We already know that, in a general Finsler manifold (M, F), the normalized element of support $l_i = \frac{\partial F}{\partial y_i}$ and the angular metric tensor h_{ij} [14] are evaluated by the following formula:

$$l_i = \frac{F_\alpha y_i}{\alpha} + F_\beta b_i \tag{2.7}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j, \tag{2.8}$$

and the coefficients are defined and calculated as follows:

$$y_i = a_{ij}y^j$$

$$p = \frac{FF_\alpha}{\alpha} = \frac{(\alpha - n\beta)(\alpha+\beta)^{2n+1}}{\alpha^{2n+2}} \tag{2.9}$$

$$q_0 = FF_{\beta\beta} = \frac{n(n+1)(\alpha+\beta)^{2n}}{\alpha^{2n}} \tag{2.10}$$

$$q_1 = \frac{FF_{\alpha\beta}}{\alpha} = -\frac{n(n+1)\beta(\alpha+\beta)^{2n}}{\alpha^{2n+2}} \tag{2.11}$$

$$\begin{aligned} q_2 &= \frac{F(F_{\alpha\alpha} - \frac{F_\alpha}{\alpha})}{\alpha^2} \\ &= \frac{(\alpha+\beta)^{2n}\left\{n\beta(n+2\beta+n\beta) - \alpha(\alpha+\beta)\right\}}{\alpha^{2n+4}} \end{aligned} \tag{2.12}$$

We also know that, in a general Finsler manifold (M, F), the fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is evaluated by [14] the following formula:

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j \tag{2.13}$$

whereas its coefficients p, p_0, p_1 and p_2 are defined and calculated as follows:

$$p = \frac{FF_\alpha}{\alpha}$$
$$= \frac{(\alpha - n\beta)(\alpha+\beta)^{2n+1}}{\alpha^{2n+2}} \tag{2.14}$$

$$p_0 = q_0 + F_\beta^2$$
$$= \frac{(n+1)(2n+1)(\alpha+\beta)^{2n}}{\alpha^{2n}} \tag{2.15}$$

$$p_1 = q_1 + \frac{pF_\beta}{F}$$
$$= \frac{(n+1)(\alpha+\beta)^{2n}(\alpha - 2n\beta)}{\alpha^{2n+2}} \tag{2.16}$$

$$p_2 = q_2 + \frac{p^2}{F^2}$$
$$= \frac{\beta(\alpha+\beta)^{2n}\left\{2n^2\beta + 2n\beta - n\alpha - \alpha\right\}}{\alpha^2(n+2)} \tag{2.17}$$

We know that, in a Finsler manifold (M, F), reciprocal metric tensor of a fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial\partial F^2}{\partial y^i \partial y^j}$ is denoted by g^{ij} and is evaluated by the formula [14]

$$g^{ij} = \frac{a^{ij}}{p} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j \tag{2.18}$$

whereas its coefficients b^i, S_0, S_1 and S_2 are evaluated by the following formulae:

$$b^i = a^{ij} b_j$$

$$S_0 = \frac{pp_0 + (p_0 p_2 - p_1^2)\alpha^2}{p\zeta} \tag{2.19}$$

$$S_1 = \frac{pp_1 + (p_0 p_2 - p_1^2)\beta}{p\zeta} \tag{2.20}$$

$$S_2 = \frac{pp_2 + (p_0 p_2 - p_1^2)b^2}{p\zeta} \tag{2.21}$$

$$\zeta = p(p + p_0 b^2 + p_1\beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2) \tag{2.22}$$

where $b^2 = a_{ij}b^i b^j$.

Let us define the hv-torsion tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ as follows [17]:

$$C_{ijk} = \frac{p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k}{2p} \tag{2.23}$$

and its coefficients γ_1 and m_i are evaluated by the formulae

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0, m_i = b_i - \frac{y_i \beta}{\alpha^2} \tag{2.24}$$

Here m_i is known as non-zero covariant vector orthogonal to element of support y^i. Let Γ^i_{jk}be the components of Christoffel symbol of the associated Riemannian space R^n and ∇_k be the covariant differentiation with respect to x^k relative to Christoffel symbol. Now we put

$$2E_{ij} = b_{ij} + b_{ji} \tag{2.25}$$

$$2F_{ij} = b_{ij} - b_{ji} \tag{2.26}$$

where $b_{ij} = \nabla_j b_i$. Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be Cartan connection of (M, F). The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of the special Finsler manifold (M, F) is given by [17]

$$\begin{aligned} D^i_{jk} =& B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k \\ & - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} Csk^m + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{2.27}$$

where

$$B_k = p_0 b_k + p_1 y_k \tag{2.28}$$

$$B^i = g^{ij} B_j \tag{2.29}$$

$$B_{ij} = \frac{p_1(a_{ij} - \frac{y_i y_j}{\alpha^2}) + \frac{\partial p_0}{\partial \beta} m_i m_j}{2} \tag{2.30}$$

$$B^k_i = g^{kj} B_{ji} \tag{2.31}$$

$$A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k \tag{2.32}$$

$$\lambda^m = B^m E_{00} + 2B_0 F^m_0 \tag{2.33}$$

$$F^k_i = g^{kj} F_{ji} \tag{2.34}$$

$$B_0 = B_i Y^i \tag{2.35}$$

Here as well as henceforward '0' denotes tensorial contraction with y^i besides p_0,q_0 and S_0.

3 Induced Cartan Connection

Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$ is generalized square metric. Also, let M^{n-1} be a hypersurface of the Finsler manifold (M, F) whose hypothetical picture is depicted in the Figure 4.

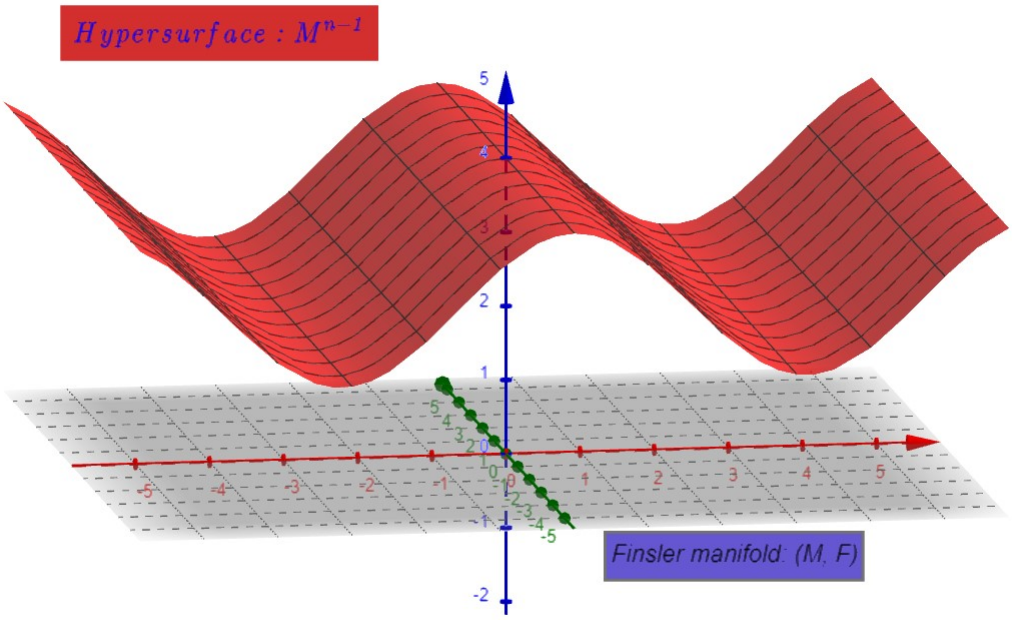

Figure 5: A hypothetical picture of hypersurface M^{n-1} of the 3-dimensional Finsler manifold (M, F).

We will describe this hypersurface M^{n-1} by following parametric equations:

$$x^i = x^i(u^\alpha), (i = 1, 2, 3, ...n; \alpha = 1, 2, 3, ...n-1) \tag{3.1}$$

where u^α is a parameter that represents coordinates on the hypersurface M^{n-1}. Do you know why we are considering parametric equation of the hypersurface? We considered parametric equation only because of the fact that parametrization always makes our life simpler. Remember those days when we used to consider parametric equations of curves and surfaces. Moreover, using parametrizations we were able to calculate speed of the curves as well as tangent planes and normal lines of the sphere. In elementary differential geometry we also have learnt that parametrization helps us to determine

the shape, that is, curvature and torsion of a curves and surfaces. Thus a parametrization has more information than the set of points constituting the hypersurface M^{n-1}.

Now differentiating the equation (3.1) of the hypersurface with respect to parameters u^α, we get $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$. Here each $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ for (α=1,2,3,.....n-1) represents components of tangent vectors and these tangent vectors B^i_α represent a tangent space at a point p of the hypersurface M^{n-1} . Let the matrix corresponding to first derivative $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ be $[B^i_\alpha] = [\frac{\partial x^i}{\partial u^\alpha}]$, and it has maximal rank , namely, (n-1). The maximal rank required here is to ensure that tangent vectors forms linearly independent set so that any generic vector tangent to M^{n-1} is linearly expressible in terms of these linearly independent tangent vectors. To introduce a Finsler structure in the hypersurface M^{n-1} , the supporting element y^i at a point u^α of M^{n-1} is assumed to be tangential to M^{n-1}, so that we may write

$$y^i = B^i_\alpha(u)v^\alpha \tag{3.2}$$

Therefore v^α is the element of support of hypersurface M^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and hv-torsion tensor $C_{\alpha\beta\gamma}$ of hypersurface M^{n-1} are defined by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma \tag{3.3}$$

Now the unit normal vector $N^i(u,v)$ at an arbitrary point u^α of the hypersurface M^{n-1} is defined as follows:

Definition 3.1. A vector $N^i(u,v)$ at a point u^α of the hypersurface M^{n-1} is said to be unit normal vector if

$$g_{ij}(x(u,v),y(u,v))B^i_\alpha N^j = 0, g_{ij}(x(u,v),y(u,v))N^iN^j = 1 \tag{3.4}$$

Let us define angular metric tensor h_{ij} as follows:

Definition 3.2. We say the tensor h_{ij} an angular metric tensor, if h_{ij} satisfies the following conditions:

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, h_{ij}B^i_\alpha N^j = 0, h_{ij}N^iN^j = 1 \tag{3.5}$$

Let (B^α_i, N_i) be the inverse of (B^i_α, N^i), then we have

$$\begin{aligned} B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, B^i_\alpha B^\beta_i = \delta^\beta_\alpha, B^\alpha_i N^i = 0, \\ B^i_\alpha N_i = 0, N_i = g_{ij}N^i, B^k_i = g^{kj}B_{ji}, \\ B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j \end{aligned}$$

We shall denote by $\mathcal{I}C\Gamma$ the connection of a hypersurface M^{n-1} induced from the Cartan connection $C\Gamma$. The induced Cartan connection $\mathcal{I}C\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^{\alpha}_{\beta}, C^{\alpha}_{\beta\gamma})$ on hypersurface M^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{i}_{jk})$ is given by [10]

$$\begin{aligned} \Gamma^{*\alpha}_{\beta\gamma} &= B^{\alpha}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}) + M^{\alpha}_{\beta}H_{\gamma} \\ G^{\alpha}_{\beta} &= B^{\alpha}_{i}(B^{i}_{0\beta} + \Gamma^{*i}_{0j}B^{j}_{\beta}) \\ C^{\alpha}_{\beta\gamma} &= B^{\alpha}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma} \end{aligned} \tag{3.6}$$

where second fundamental v-tensor $M_{\beta\gamma}$ is defined by

$$\begin{aligned} M_{\beta\gamma} &= N_i C^i_{jk} B^j_\beta B^k_\gamma \\ &= N_i g^{li} C_{ljk} B^j_\beta B^k_\gamma \quad (\text{using } C^i_{jk} = g^{li} C_{ljk}) \\ &= N^l C_{ljk} B^j_\beta B^k_\gamma \quad (\text{using } N^i g^{li} = N^l) \\ &= C_{ijk} B^i_\beta B^j_\gamma N^k \quad (\text{adjusting the indices } j,\ k \text{ and } l) \\ M^\alpha_\beta &= g^{\alpha\gamma} M_{\beta\gamma} \end{aligned} \tag{3.7}$$

and normal curvature vector H_β is defined by

$$H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta),$$

where

$$\begin{aligned} B^i_{\beta\gamma} &= \frac{\partial B^i_\beta}{\partial U^\gamma} \\ B^i_{0\beta} &= B^i_{\alpha\beta} v^\alpha \end{aligned}$$

The quantities $M_{\beta\gamma}$ and H_β appeared in above equations are called the second fundamental v-tensor and normal curvature vector respectively [10]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [10]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma \tag{3.8}$$

where

$$M_\beta = C^i_{jk}B^j_\beta N_i N^k = C_{ijk}B^i_\beta N^j N^k \tag{3.9}$$

The relative h-covariant derivative and v-covariant derivative of projection factor B^i_α with respect to induced Cartan connection $\mathcal{I}C\Gamma$ are respectively given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i \tag{3.10}$$

$$B^i_{\alpha|\beta} = M_{\alpha\beta}N^i \tag{3.11}$$

The equation (3.8) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta \tag{3.12}$$

Thus the above equation simplifies to

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0 \tag{3.13}$$

Definition 3.3 ([10]). A hypersurface M^{n-1} of a Finsler manifold (M, F) is said to be a hyperplane of first kind if each path connecting two different points of the hypersurface M^{n-1} with respect to the induced Cartan connection $\mathcal{I}C\Gamma$ is also becomes the path of the ambient Finsler manifold (M, F) with respect to Cartan connection $C\Gamma$.

Definition 3.4 ([10]). A hypersurface M^{n-1} of a Finsler manifold (M, F) is said to be a hyperplane of second kind if each h-path of the hypersurface M^{n-1} with respect to the induced Cartan connection $\mathcal{I}C\Gamma$ is also the h-path of the ambient Finsler manifold (M, F) with respect to Cartan connection $C\Gamma$.

Definition 3.5 ([10]). A hypersurface M^{n-1} of a Finsler manifold (M, F) is said to be a hyperplane of third kind if unit normal vector B^i of the hypersurface M^{n-1} with respect to the metric F is parallel along each curve (u^α, v^α).

If one wants to prove a hypersurface a hyperplane of first kind, hyperplane of second kind and hyperplane of third kind, it is very difficult to prove it only with the help of definitions mentioned above, in that situation one should incorporate below sufficient conditions given by M. Matsumoto to prove the same:

Lemma 3.6 ([10]). *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector H_β vanishes.*

Lemma 3.7 ([10]). *A hypersurface $M^{n-1}$is a hyperplane of first kind if and only if $H_\alpha = 0$.*

Lemma 3.8 ([10]). *A hypersurface M^{n-1} is a hyperplane of second kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3.9 ([10]). *A hypersurface M^{n-1} is a hyperplane of third kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

4 Hypersurface M^{n-1} of the special Finsler space

We know that a hypersurface is an embedded subspace of codimension 1. That is, a subspace of dimension less that 1 than the dimension of a given manifold is called hypersuraface of the given manifold or space. For the sake of our study we are dealing with hypersurfaces rather than the subspaces of arbitrary dimensions. In this paper we are specifically confined to Finslerian hypersurfaces M^{n-1}.
Let us proof the following propositions in context of Finslerian hypersurfaces.

Proposition 4.1. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its hypersurface. Then fundamental function of the hypersurface M^{n-1} induced from the Finsler manifold (M, F) is a Riemannian metric.*

Proof. It is given that (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric. Let level equation of the hypersurface M^{n-1} be given by

$$b(x) = c$$

where c is a real number.
Take the gradient of the above level equation representing hypersurface M^{n-1}, we get

$$b_i(x) = \partial_i b$$

Again consider the parametric equation of the same hypersurface M^{n-1} as

$$x^i = x^i(u^\alpha)$$

Differentiating the equation of hypersurface $b(x(u)) = c$ with respect to parameter u^α, we get

$$\frac{\partial b(x(u))}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0$$
$$b_i(x)B^i_\alpha = 0$$

where $b_i(x) = \frac{\partial b(x(u))}{\partial x^i}$ and $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$

This implies that $b_i(x)$ are normal vector field (covariant component) of hypersurface M^{n-1}.

Thus at any point of the hypersurface M^{n-1} we now have

$$b_i B^i_\alpha = 0 \tag{4.1}$$

$$b_i y^i = 0, i.e., \beta = 0 \tag{4.2}$$

Now, we will see how generalized square metric $F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, induces a metric on the hypersurface M^{n-1}. In this case we will denote induced metric by $\bar{F}$. First consider the generalized square metric

$$\begin{aligned}
F &= \frac{(\alpha+\beta)^{n+1}}{\alpha^n} \\
&= \frac{(\sqrt{a_{ij}y^iy^j} + b_iy^i)^{n+1}}{(\sqrt{a_{ij}y^iy^j})^n} \\
&= \frac{\left(\sqrt{a_{ij}B^i_\alpha(u)v^\alpha B^j_\beta(u)v^\beta} + b_iy^i\right)^{n+1}}{\left(\sqrt{a_{ij}B^i_\alpha(u)v^\alpha B^j_\beta(u)v^\beta}\right)^n} \\
&= \frac{\left(\sqrt{a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta} + b_iy^i\right)^{n+1}}{\left(\sqrt{a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta}\right)^n}
\end{aligned}$$

which is the general induced metric on the corresponding hypersurface M^{n-1}. Using equation (4.2), general induced metric of the hypersurface becomes

$$\begin{aligned}
F(u,v) &= \frac{\left(a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta\right)^{\frac{n+1}{2}}}{\left(a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta\right)^{\frac{n}{2}}} \\
&= \sqrt{a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta} \\
\bar{F}(u,v) &= \sqrt{a_{\alpha\beta}v^\alpha v^\beta}
\end{aligned} \tag{4.3}$$

where $a_{\alpha\beta} = a_{ij}B^i_\alpha(u)B^j_\beta(u)$.
Thus function represented by equation (4.3) is fundamental function or the metric of the hypersurface M^{n-1} induced from the ambient Finsler manifold (M, F).

The fundamental function of the hypersurface M^{n-1} represented by equation (4.3) do not have β component as $\beta = b_iy^i = 0$ over the hypersurface M^{n-1} therefore fundamental function of the hypersurface M^{n-1} induced from the Finsler manifold (M, F) is a Riemannian metric. □

Proposition 4.2. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the covariant and contravariant components of normal vector field on the hypersurface M^{n-1} are given by*

1. $b_i = \sqrt{\frac{b^2}{1+n(n+1)}} N_i$

2. $b^i = \sqrt{b^2\{1+n(n+1)\}} N^i + \frac{b^2}{\alpha} y^i$

Proof. It is given that M^{n-1} is a hypersurface of the manifold (M, F), where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric.

Moreover, we know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1}. Let us calculate the value of p, p_0, p_1 and p_2. For that, substitute the value of $\beta = 0$ into equations (2.14), (2.15), (2.16), and (2.17), we get

$$p = 1, p_0 = (n+1)(2n+1), p_1 = \frac{n+1}{\alpha}, p_2 = 0 \tag{4.4}$$

Now put the values of p, p_0, p_1, p_2 into equations (2.19), (2.20), (2.21) and (2.22), we get

$$S_0 = \frac{n(n+1)}{1+n(n+1)b^2} \tag{4.5}$$

$$S_1 = \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \tag{4.6}$$

$$S_2 = -\frac{(n+1)^2 b^2}{\alpha^2\{1+n(n+1)b^2\}} \tag{4.7}$$

$$\zeta = 1+n(n+1)b^2 \tag{4.8}$$

Substituting the values of p, S_0, S_1, S_2 from the equations (4.4), (4.5), (4.6) and (4.7) into equation (2.18), we have

$$g^{ij} = \frac{a^{ij}}{1} - \frac{n(n+1)}{1+n(n+1)b^2} \times b^i b^j - \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \times (b^i y^j + b^j y^i) + \frac{(n+1)^2 b^2}{\alpha^2\{1+n(n+1)b^2\}} \times y^i y^j \tag{4.9}$$

Multiplying equation 4.9 by $b_i b_j$ and using $\beta = b_i y^i = 0$, over the hyper-

surface M^{n-1}, it becomes

$$g^{ij} = a^{ij} \times b_i b_j - \frac{n(n+1)}{1+n(n+1)b^2} \times b^i b^j \times b_i b_j - \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \times (b^i y^j + b^j y^i) \times b_i b_j + \frac{(n+1)^2 b^2}{\alpha^2\{1+n(n+1)b^2\}} \times y^i y^j \times b_i b_j$$

$$= (a^{ij} b_i) b_j - \frac{n(n+1)}{1+n(n+1)b^2} \times (b^i b_i)(b^j b_j) - \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \times \{(b^i b_i)(b_j y^j) + (b_i b^j)(b_i y^i)\} + \frac{(n+1)^2 b^2}{\alpha^2\{1+n(n+1)b^2\}} \times \{(b_i y^i)(b_j y^j)\}$$

$$= b^j b_j - \frac{n(n+1)}{1+n(n+1)b^2} \times (b^2)(b^2) - \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \times \{(b^2)(0) + (b^2)(0)\} + \frac{(n+1)^2 b^2}{\alpha^2\{1+n(n+1)b^2\}} \times \{(0)(0)\}$$

$$= b^2 - \frac{n(n+1)}{1+n(n+1)b^2} \times b^4$$

$$g^{ij} b_i b_j = \frac{b^2}{1+n(n+1)}$$

Thus at any generic point of the hypersurface M^{n-1}, we have

$$g^{ij} b_i b_j = \frac{b^2}{1+n(n+1)}$$

Now from the above equation and using the equation (3.4), we get

$$b_i = \sqrt{\frac{b^2}{1+n(n+1)}} N_i \tag{4.10}$$

which is the covariant component of the normal vector field on the hypersurface M^{n-1}. Now from (4.9) and (4.10) we get

$$b^i = a^{ij} b_j = \sqrt{b^2\{1+n(n+1)\}} N^i + \frac{b^2}{\alpha} y^i \tag{4.11}$$

which is the contravariant component of the normal vector field on the hypersurface M^{n-1}. □

Proposition 4.3. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then second fundamental v-tensor of hypersurface M^{n-1} is given by*

$$M_{\alpha\beta} = \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right) h_{\alpha\beta}$$

and second fundamental h-tensor $H_{\alpha\beta}$ is symmetric, i.e., $H_{\alpha\beta} = H_{\beta\alpha}$.

Proof. It is given that M^{n-1} is a hypersurface of the manifold (M, F), where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric.

Moreover, we know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1}. Put the value of $\beta = 0$ into equations (2.14), (2.15), (2.16), and (2.17), we get

$$p = 1, p_0 = (n+1)(2n+1), p_1 = \frac{n+1}{\alpha}, p_2 = 0$$

Now, put the values of p, p_0, p_1 and p_2 obtained above into equation 2.13, we get fundamental metric tensor of the hypersurace M^{n-1}

$$g_{ij} = a_{ij} + (n+1)(2n+1)b_i b_j + \frac{(n+1)}{\alpha}(b_i y_j + b_j y_i) \tag{4.12}$$

Let us calculate the value of q_0, q_1 and q_2. For that, substitute the value of $\beta = 0$ into equations (2.10), (2.11), and (2.12), we get

$$q_0 = n(n+1), q_1 = 0, q_2 = -\frac{1}{\alpha^2}$$

Substituting the values of p, q_0, q_1 and q_2 in equation 2.8, we get angular metric tensor of the hypersurface M^{n-1}

$$h_{ij} = a_{ij} + n(n+1)b_i b_j - \frac{1}{\alpha^2} y_i y_j \tag{4.13}$$

Differentiating equation 2.15 with respect to β, we have

$$\frac{\partial p_0}{\partial \beta} = \frac{2n(n+1)(2n+1)(\alpha+\beta)^{2n-1}}{\alpha^{2n}}.$$

We know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1} so put the value of $\beta = 0$ into above equation and equation 2.24, we get

$$\begin{aligned} \frac{\partial p_0}{\partial \beta} &= \frac{2n(n+1)(2n+1)}{\alpha} \\ \gamma_1 &= \frac{n(n^2-1)}{\alpha} \\ m_i &= b_i \end{aligned}$$

Using the values of p, p_1, γ_1 and m_i in equation 2.23, hv-torsion tensor on the hypersurface M^{n-1}, becomes

$$C_{ijk} = \frac{(n+1)\left[(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + n(n^2-1)b_ib_jb_k\right]}{2\alpha} \tag{4.14}$$

Substituting the value of C_{ijk} from equation 4.14 in equation 3.7 as follows:

$$M_{\beta\gamma} = C_{ijk}B^i_\beta B^j_\gamma N^k$$
$$\therefore M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k$$
$$= \left[\frac{(n+1)\left(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right) + n(n^2-1)b_ib_jb_k\right]}{2\alpha}\right] B^i_\alpha B^j_\beta N^k$$
$$= \left[\frac{(n+1)\left\{h_{ij}B^i_\alpha B^j_\beta b_k + h_{jk}B^j_\beta(b_iB^i_\alpha) + h_{ki}B^j_\alpha(b_jB^j_\beta)\right\} + n(n^2-1)(b_iB^i_\alpha)(b_jB^j_\beta)b_k}{2\alpha}\right] N^k$$

Using the equation 4.1 in above expression, we get

$$= \left[\frac{(n+1)\left\{h_{ij}B^i_\alpha B^j_\beta b_k + h_{jk}B^j_\beta(0) + h_{ki}B^j_\alpha(0)\right\} + n(n^2-1)(0)(0)b_k}{2\alpha}\right] N^k$$
$$= \left[\frac{(n+1)h_{ij}B^i_\alpha B^j_\beta b_k}{2\alpha}\right] N^k$$
$$= \left[\frac{(n+1)(h_{ij}B^i_\alpha B^j_\beta)b_kN^k}{2\alpha}\right]$$

Using the equation 3.5, we get

$$= \left[\frac{(n+1)h_{\alpha\beta}b_kN^k}{2\alpha}\right]$$

Using the equation 4.10 in above expression, we get

$$= \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}N_k\right)h_{\alpha\beta}N^k$$
$$= \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right)h_{\alpha\beta}N_kN^k$$

We know that $N_k N^k = 1$. Use this fact in above expression, we get

$$M_{\alpha\beta} = \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right) h_{\alpha\beta} \tag{4.15}$$

Again, substituting the value of C_{ijk} from equation 4.14 into equation 3.9 as follows:

$$\begin{aligned} M_\beta &= C_{ijk} B^i_\beta N^j N^k \\ \therefore M_\alpha &= C_{ijk} B^i_\alpha N^j N^k \\ &= \left[\frac{(n+1)\left(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right) + (n^2-1)b_i b_j b_k\big]}{2\alpha}\right] B^i_\alpha N^j N^k \end{aligned}$$

$$= \left[\frac{(n+1)\left(h_{ij}b_k B^i_\alpha N^j N^k + h_{jk}b_i B^i_\alpha N^j N^k + h_{ki}b_j B^i_\alpha N^j N^k\right) + n(n^2-1)b_i B^i_\alpha b_j b_k N^j N^k\big]}{2\alpha}\right]$$

Using equations 4.1 and 3.5 in above expression, we get

$$M_\alpha = 0 \tag{4.16}$$

Substituting the value of M_α from the equation 4.16 in equation 3.12 as follows:

$$\begin{aligned} H_{\beta\gamma} - H_{\gamma\beta} &= M_\beta H_\gamma - M_\gamma H_\beta \\ H_{\beta\gamma} - H_{\gamma\beta} &= 0 \times H_\gamma - 0 \times H_\beta \\ H_{\beta\gamma} - H_{\gamma\beta} &= 0 \\ H_{\beta\gamma} &= H_{\gamma\beta} \\ \therefore H_{\alpha\beta} &= H_{\beta\gamma} \end{aligned}$$

which shows that $H_{\alpha\beta}$ is symmetric. □

Theorem 4.4. *Let (M, F) be a Finsler manifold, where $F(\alpha,\beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the hypersurface M^{n-1} will be hyperplane of first kind if and only if*

$$2b_{ij} = b_i c_j + b_j c_i$$

Moreover we show that second fundamental tensor $H_{\alpha\beta}$ of M^{n-1} is proportional to it's angular metric tensor $h_{\alpha\beta}$. That is, $H_{\alpha\beta} = \frac{c_0 b}{\sqrt{1+n(n+1)}} h_{\alpha\beta}$.

Proof. Let us differentiate equation 4.1 with respect to β, we get

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0 \tag{4.17}$$

Put the value of $B^i_{\alpha|\beta}$ from equation 3.10 and $b_{i|\beta} = b_{i|j}B^j_\beta + b_{i|j}N^j H_\beta$ into equation 4.17, we get

$$b_{i|j}B^j_\beta B^i_\alpha + b_{i|j}N^j H_\beta B^i_\alpha + b_i H_{\alpha\beta}N^i = 0 \tag{4.18}$$

We know that

$$b_{i|j} = -b_h C^h_{ij}$$

Put the value of b_h from equation 4.10 into above expression as follows:

$$\begin{aligned}
b_{i|j} &= -b_h C^h_{ij} \\
b_{i|j} &= -\sqrt{\frac{b^2}{1+n(n+1)}}N_h C^h_{ij} \\
b_{i|j}B^i_\alpha N^j &= -\sqrt{\frac{b^2}{1+n(n+1)}}N_h C^h_{ij}B^i_\alpha N^j \\
&= -\sqrt{\frac{b^2}{1+n(n+1)}}M_\alpha \\
&= -\sqrt{\frac{b^2}{1+n(n+1)}} \times 0 \quad \text{(using Equation (4.16))} \\
&= 0
\end{aligned}$$

Using $b_{i|j}B^i_\alpha N^j = 0$ and equation 4.10 in the equation 4.18 and then using the fact that $N_i N^i = 1$, we get

$$b_{i|j}B^j_\beta B^i_\alpha + \sqrt{\frac{b^2}{1+n(n+1)}}H_{\alpha\beta} = 0 \tag{4.19}$$

It is obvious that $b_{i|j}$ is symmetric. Now contracting 4.19 with v^β first and then with v^α respectively and using the equations 3.2, 3.13 and 4.16, we get

$$b_{i|j}B^i_\alpha y^j + \sqrt{\frac{b^2}{1+n(n+1)}}H_\alpha = 0 \tag{4.20}$$

$$b_{i|j}y^i y^j + \sqrt{\frac{b^2}{1+n(n+1)}}H_0 = 0 \tag{4.21}$$

We know from the Lemma 3.6 and Lemma 3.7, a hypersurface M^{n-1} is a hyperplane of first kind if and only if normal curvature vanishes, i.e., $H_0 = 0$. Using the value $H_0 = 0$ in equation 4.21 we find that hypersurface M^{n-1} is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. This $b_{i|j}$ is the covariant derivative of with respect to Cartan connection $C\Gamma$ of Finsler space F, it may depend on y^i. Moreover $\nabla_j b_i = b_{ij}$ is the covariant derivative of b_i with respect to Riemannian connection Γ^i_{jk} constructed from $a_{ij}(x)$, therefore b_{ij} dose not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ of above covariant derivatives in further discussion. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ is given by equation 2.27. Since b_i is a gradient vector, from equations 2.25 and 2.26 we have

$$E_{ij} = b_{ij}, F_{ij} = 0, F^i_j = 0 \tag{4.22}$$

Using equation 4.22 into equation 2.27, we get

$$\begin{aligned} D^i_{jk} = {} & b_{jk}B^i + b_{0k}B^i_j + b_{0j}B^i_k - b_{0m}g^{im}B_{jk} \\ & - A^m_k C^i_{jm} - A^m_j C^i_{km} + A^m_s C_{jkm}g^{is} \\ & + \lambda^s(C^m_{sk}C^i_{jm} + C^m_{sj}C^i_{km} - C^i_{ms}C^m_{jk}) \end{aligned} \tag{4.23}$$

Using the equations 4.2, 4.4, 4.5 and 4.6 into equations 2.28 to 2.23, we get

$$B_k = (n+1)(2n+1)b_k + \frac{n+1}{\alpha}y_k, B^i = bb^i + by^i \tag{4.24}$$

$$B_{ij} = \frac{(n+1)\left\{a_{ij}\alpha^2 - y_i y_j + 2n(n+1)b_i b_j \alpha\right\}}{2\alpha^3} \tag{4.25}$$

$$B^i_j = 0 \tag{4.26}$$

$$A^m_k = 0, \lambda^m = B^m b_{00}. \tag{4.27}$$

Using tensor contraction operation with equations 4.25 and 4.26 by y^j, we get $B_{i0} = 0$, $B^i_0 = 0$. Further contracting equation 4.27 by y^k and using the fact that $B^i_0 = 0$, we get $A^m_0 = B^m b_{00}$. Contracting equation 4.23 by y^k and using the facts $B_{i0} = 0$, $B^i_0 = 0$, $A^m_0 = B^m b_{00}$ and $C^m_{s0} = 0$, $C^i_{0m} = 0$, $C^m_{j0} = 0$ obtained by contracting equations 4.25, 4.26, 4.27 and 3.6, we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - b_{00}B^m C^i_{jmi} \tag{4.28}$$

$$D^i_{00} = bb^i b_{00} + by^i b_{00} \tag{4.29}$$

Multiplying equation 4.25 by b_i and then using equations 4.2, 4.21 and 4.23, we get

$$b_i D^i_{j0} = bb_{j0} + bb_j b_{00} - bb_i b^m C^i_{jm} b_{00} \tag{4.30}$$

Now multiplying equation 4.26 by b_i and then using equation 4.2 we get

$$b_i D^i_{00} = \frac{b^2}{1+n(n+1)} b_{00} \tag{4.31}$$

From equations 4.14 and 4.16 it is clear that

$$b^m b_i C^i_{jm} B^j_\alpha = \frac{b^2}{1+n(n+1)} M_\alpha = 0 \tag{4.32}$$

Contracting the expression $b_{i|j} = b_{ij} - b_r D^r_{ij}$ by y^i and y^j respectively and then using equation 4.31 we get

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{b^2}{1+n(n+1)} b_{00}$$

Put $b_{i|j} = b_{ij} - b_r D^r_{ij}$ in equations 4.17 and 4.18 and then using equations 4.27, 4.1 and 4.29 and the value of $b_{i|j} y^i y^j$ above, equations 4.20 and 4.21 can be written as

$$\sqrt{\frac{b^2}{1+n(n+1)}} b_{i0} B^i_\alpha + b H_\alpha = 0 \tag{4.33}$$

$$\sqrt{\frac{b^2}{1+n(n+1)}} b_{00} + b H_0 = 0 \tag{4.34}$$

From the equation 4.31 it is clear that the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} is independent of y^i. Since y^i satisfy equation 4.2, the condition can be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i \tag{4.35}$$

Thus we shown that a Finslerian hypersurface M^{n-1} will be hyperplane of first kind if and only if $2b_{ij} = b_i c_j + b_j c_i$.

Now we try to show that second fundamental tensor $H_{\alpha\beta}$ of the hypersurface M^{n-1} is proportional to its angular metric tensor $h_{\alpha\beta}$.

For that, contracting equation 4.35 and using the fact that $b_i y^j = 0$, we get $b_{00} = 0$. This implies that the condition $b_{00} = 0$ and $2b_{ij} = b_i c_j + b_j c_i$ are equivalent.

Multiplying equation 4.35 by B^i_α and then B^j_β and using equations (4.1)

and (4.2), we have

$$\begin{aligned}
2b_{ij} \times B^i_\alpha &= \left(b_i B^i_\alpha\right) c_j + b_j c_i B^i_\alpha \\
2b_{ij} B^i_\alpha &= 0 \times c_j + b_j c_i B^i_\alpha \\
2b_{ij} B^i_\alpha \times B^j_\beta &= \left(b_j B^j_\beta\right) c_i B^i_\alpha \\
2b_{ij} B^i_\alpha B^j_\beta &= 0 \times c_i B^i_\alpha \\
2b_{ij} B^i_\alpha B^j_\beta &= 0 \\
b_{ij} B^i_\alpha B^j_\beta &= 0
\end{aligned}$$

Again, multiplying equation 4.35 by B^i_α and y^j and then using Equation (4.1), we have

$$\begin{aligned}
2b_{ij} \times B^i_\alpha y^j &= \left(b_i B^i_\alpha\right) c_j y^j + \left(b_j y^j\right) c_i B^i_\alpha \\
2b_{ij} B^i_\alpha y^j &= 0 \times c_j y^j + 0 \times c_i B^i_\alpha \\
2b_{ij} B^i_\alpha y^j &= 0 \\
b_{ij} B^i_\alpha y^j &= 0 \\
b_{i0} B^i_\alpha &= 0 \quad \text{(contraction by } y^j \text{ is taking place)}
\end{aligned}$$

Again, consider equation (4.35)

$$\begin{aligned}
2b_{ij} &= b_i c_j + b_j c_i \\
2b_{ij} y^j &= b_i \left(c_j y^j\right) + \left(b_j y^j\right) c_i \quad \text{(multiplying by } y^j \text{ both sides)} \\
2b_{i0} &= b_i c_0 + 0 \times c_i \quad \text{(contraction by } y^j \text{ is taking place)} \\
2b_{i0} &= b_i c_0 \\
2b_{i0} b^i &= \left(b_i b^i\right) c_0 \quad \text{(multiplying by } b^i \text{ both sides)} \\
2b_{i0} b^i &= b^2 c_0 \quad (\because b^2 = b_i b^i) \\
b_{i0} b^i &= \frac{b^2 c_0}{2}
\end{aligned}$$

Using this in equation 4.30 gives $H_\alpha = 0$. Now using 4.23 and 4.24 and using $b_{00} = 0$ and $b_{ij} B^i_\alpha B^j_\beta = 0$, we get $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}$. Thus using the equations 4.6, 4.7, 4.8, 4.12 and 4.20, we get

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2}{1 + n(n+1)} h_{\alpha\beta} \tag{4.36}$$

Thus using the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equation 4.36, equation 4.19 reduces to

$$-\frac{c_0 b^2}{1 + n(n+1)} h_{\alpha\beta} + \sqrt{\frac{b^2}{1 + n(n+1)}} H_{\alpha\beta} = 0 \tag{4.37}$$

$$H_{\alpha\beta} = \frac{c_0 b}{\sqrt{1+n(n+1)}} h_{\alpha\beta}$$

Thus we shown that the second fundamental tensor $H_{\alpha\beta}$ of M^{n-1} is proportional to its angular metric tensor $h_{\alpha\beta}$.

□

Theorem 4.5. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface.Then the hypersurface M^{n-1} will be hyperplane of second kind if and only if*

$$b_{ij} = e b_i b_j$$

Proof. We know from Lemma 3.8, hypersurface M^{n-1} is a hyperplane of second kind if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Now we consider these two sufficient conditions one by one.

1. If $H_{\alpha\beta} = 0$, then equation (4.37) becomes

$$\begin{aligned} -\frac{c_0 b^2}{1+n(n+1)} h_{\alpha\beta} + \sqrt{\frac{b^2}{1+n(n+1)}} \times 0 = 0 \\ \implies -\frac{c_0 b^2}{1+n(n+1)} h_{\alpha\beta} = 0 \\ \implies c_0 = 0 \\ \implies c_0 = c_i y^i = 0 \end{aligned}$$

   $\implies$ there exist a function $e(x)$ such that $c_i(x) = e(x) b_i(x)$ and this $c_i(x) = e(x) b_i(x)$ forces c_0 to vanish as follows:

$$\begin{aligned} c_0 &= c_i(x) y^i \\ &= e(x) b_i(x) y^i \\ &= e(x) \left(b_i(x) y^i \right) \\ &= e(x) \times 0 \quad \text{(as along the hypersurface } b_i y^i = 0) \\ \therefore c_0 &= 0 \end{aligned}$$

2. Again, if $H_\alpha = 0$, then Lemma 3.6 and Lemma 3.7 imply $H_0 = 0$. We have already shown above $H_0 = 0$ is equivalent to

$$2 b_{ij} = b_i c_j + b_j c_i$$

Now we combine case 1 and case 2. For that, put the value of $c_i(x) = e(x)b_i(x)$ obtained in case 1 to equation obtained in case 2, we get

$$\begin{aligned} 2b_{ij} &= b_i e(x) b_j(x) + b_j e(x) b_i(x) \\ &= e\left(b_i b_j + b_j b_i\right) \\ &= e \times 2 b_i b_j \\ \therefore 2b_{ij} &= 2e b_i b_j \\ \implies b_{ij} &= e b_i b_j \end{aligned} \tag{4.38}$$

Thus we shown that the hypersurface M^{n-1} of the Finsler manifold (M, F) will be hyperplane of second kind iff $b_{ij} = eb_ib_j$.

□

Theorem 4.6. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the hypersurface M^{n-1} will not be hyperplane of third kind.*

Proof. We know from sufficient conditions of Lemma 3.9 a hypersurface becomes a hyperplane of third kind if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$. Now we consider these three sufficient conditions one by one.

1. If $H_\alpha = 0$, then we get the condition $2b_{ij} = b_i e(x) b_j(x) + b_j e(x) b_i(x)$, which has already been proved above and is termed as the condition of hyperplane of first kind.

2. If $H_{\alpha\beta} = 0$, then we get the condition $b_{ij} = eb_ib_j$, which has already been proved above and is termed as the condition of hyperplane of second kind.

3. Now put $M_{\alpha\beta} = 0$ in Equation (4.15), we get

$$0 = \frac{(n+1)}{2\alpha} \left(\sqrt{\frac{b^2}{1 + n(n+1)}} \right) h_{\alpha\beta}$$

   which implies that no condition could be deduced to satisfy $M_{\alpha\beta} = 0$, i.e., it is impossible to find a condition under which a hypersurface becomes a hyperplane of third kind, as term on the R.H.S. of the above equation can never be zero.

Finally, we shown that hypersurface M^{n-1} is not a hyperplane of third kind. □

Corollary 4.7. *Let (M, F) be a Finsler manifold, where F may be any of the following Finsler metrics obtained by generalized square metric $F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n = 1, 2, 3, ...$*

1. $F = \frac{(\alpha+\beta)^2}{\alpha}$ *(popularly known as square metric)*

2. $F = \frac{(\alpha+\beta)^3}{\alpha^2}$

3. $F = \frac{(\alpha+\beta)^4}{\alpha^3}$, *etc.,*
Also let M^{n-1} be the corresponding hypersurfaces of the given Finsler manifold (M, F). Then, in either case, show that the hypersurface is a hyperplane of first kind, second kind and not of the third kind.

Proof. By Theorems 4.4, 4.5 and 4.6 it can be easily deduce that the hypersurfaces M^{n-1} corresponding to different Finsler manifolds (M, F) are a hyperplane of first kind, second kind and not of the third kind. It is remarkable that corresponding author has already been published a paper [13] on part (1) of this corollary. □

Conclusion. *Now after all, one may ask why authors of the article is so interested to carry forward the theory of hypersurface over Finsler space with generalized square metric $F(x, y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$, where $n = 1, 2, 3, ...$ My answer is very assertive that generalization of any theory is always fascinating due to its nature to bring various special cases under one umbrella. For example, the theorems that we have proved works for every natural number $n \in N$.*

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